the product of initial test-section pressure and ionization time as a function of reflected-shock-region temperature. It illustrates the close agreement between results obtained from Kuiper's interferometric measurements of the thermal boundary layer and those obtained from the present heat-gage measurements on the wall.

Measurements of the second rise show that ionization relaxation increases the thermal transport to the wall by a significant amount. For example, heat transfer was approximately doubled for the case of an incident-shock Mach number of 8 advancing into room-temperature argon at 10 torr pressure. The second rise of the heat-gage response levels off to a fairly constant value. That should permit some future deduction of thermal conductivity coefficients in ionized gases as well.

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Thermal Stresses in a Transversely Isotropic, Hollow, Circular Cylinder

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Introduction

FORMULAS for the thermal stresses in an isotropic, hollow, circular cylinder with a two-dimensional temperature distribution independent of the axial coordinate have been derived by Forray.[1–3] In this Note, formulas are developed for the thermal stresses in a transversely isotropic, hollow, circular cylinder. These results can be used to estimate thermal stresses in re-entry vehicles which arise from aerodynamic heating.

Statement of the Problem

Utilizing the static, linear theory of thermoelasticity for a state of plane strain, expressions are developed for the stress and displacement fields in a hollow, circular cylinder of inner radius a and outer radius b which is fabricated from a material exhibiting transverse isotropy along the radial direction. The cylinder is subjected to an arbitrary temperature distribution independent of the axial coordinate and the cylindrical surfaces are assumed stress free.

Under conditions of plane strain, the stress-displacement relations in cylindrical coordinates are

$$\sigma_r = c_{11}(\partial U/\partial R) + c_{12}[(U/R) + (\partial V/R\partial\theta)] - \lambda_1 T \quad (1a)$$

$$\sigma_\theta = c_{12}(\partial U/\partial R) + c_{22}[(U/R) + (\partial V/R\partial\theta)] - \lambda_2 T \quad (1b)$$

$$\sigma_z = c_{12}(\partial U/\partial R) + c_{23}[(U/R) + (\partial V/R\partial\theta)] - \lambda_2 T \quad (1c)$$

$$\tau_{r\theta} = c_{44}[(\partial U/R\partial\theta) + (\partial V/\partial R) - (V/R)] \quad (1d)$$

where

$$\lambda_1 = c_{11}\alpha_1 + 2c_{12}\alpha_2, \qquad \lambda_2 = c_{12}\alpha_1 + (c_{22} + c_{23})\alpha_2 \quad (1e)$$

and

$$U = u/b, \qquad V = v/b, \qquad R = r/b \quad (1f)$$

In Eqs. (1), the c_{ij} are elastic constants‡; α_1 and α_2 are coefficients of thermal expansion in the radial and circumferential directions, respectively; T is the temperature measured relative to an ambient temperature at which the body is stress free; u, v are the radial and circumferential components of displacements, respectively. Equilibrium requires that the dimensionless displacement components U and V satisfy the system of equations

$$c_{11}\frac{\partial^2 U}{\partial R^2} + \frac{c_{44}}{R^2}\frac{\partial^2 U}{\partial\theta^2} + \frac{c_{11}}{R}\frac{\partial U}{\partial R} - \frac{c_{22}}{R^2}U + \frac{(c_{12}+c_{44})}{R}\frac{\partial^2 V}{\partial R\partial\theta} - \frac{(c_{22}+c_{44})}{R^2}\frac{\partial V}{\partial\theta} = \lambda_1\frac{\partial T}{\partial R} + \frac{(\lambda_1-\lambda_2)}{R}T \quad (2a)$$

$$c_{44}\frac{\partial^2 V}{\partial R^2} + \frac{c_{22}}{R^2}\frac{\partial^2 V}{\partial\theta^2} + \frac{c_{44}}{R}\frac{\partial V}{\partial R} - \frac{c_{44}}{R^2}V + \frac{(c_{12}+c_{44})}{R}\frac{\partial^2 U}{\partial R\partial\theta} + \frac{(c_{22}+c_{44})}{R^2}\frac{\partial U}{\partial\theta} = \frac{\lambda_2}{R}\frac{\partial T}{\partial\theta} \quad (2b)$$

for some prescribed temperature T. For this problem, T will be assumed symmetric about $\theta = 0$ and to be expressible in the form

$$T(R,\theta) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} T_{nm}R^m \cos n\theta \quad (3)$$

The stress and displacement fields associated with this particular form of the temperature T have representations

$$U = \sum_{n=0}^{\infty} U_n(R)\cos n\theta, \qquad V = \sum_{n=1}^{\infty} V_n(R)\sin n\theta \quad (4a)$$

$$\sigma_r = \sum_{n=0}^{\infty} S_r^n(R)\cos n\theta, \qquad \sigma_\theta = \sum_{n=0}^{\infty} S_\theta^n(R)\cos n\theta$$

$$\tau_{r\theta} = \sum_{n=1}^{\infty} S_{r\theta}^n(R)\sin n\theta \quad (4b)$$

Stress Formulas

Expressions will now be developed for the R-dependent terms $U_n(R)$, $V_n(R)$, $S_r^n(R)$, $S_\theta^n(R)$, and $S_{r\theta}^n(R)$ of Eqs. (4). In order to circumvent certain degeneracies which occur in the field Eqs. (2) for $n = 0,1$, these values of n will be considered separately from the general case $n \geq 2$.

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‡ Relationships between the c_{ij} and Young's moduli and Poisson's ratios are given in Ref. 4.

Solution for n = 0

This is the symmetric problem for which $V_0(R) = 0$ and the field Eqs. (2) for this case reduce to

$$c_{11}\frac{\partial^2 U_0}{\partial R^2} + \frac{c_{11}}{R}\frac{\partial U_0}{\partial R} - \frac{c_{22}}{R^2}U_0 = \sum_{m=0}^{\infty}[(m+1)\lambda_1 - \lambda_2]T_{0m}R^{m-1} \tag{5}$$

which has the solution

$$U_0(R) = A_{01}R^{P_{01}} + A_{02}R^{P_{02}} + \sum_{m=0}^{\infty}\left[\frac{(m+1)\lambda_1 - \lambda_2}{(m+1)^2c_{11} - c_{22}}\right]T_{0m}R^{m+1} \tag{6a}$$

where

$$P_{01} = -P_{02} = (c_{22}/c_{11})^{1/2} \tag{6b}$$

and A_{01}, A_{02} are arbitrary constants to be determined from the traction-free boundary conditions.

The stress-displacement relations of Eqs. (1) together with the traction-free boundary conditions give

$$S_r{}^0(R) = \sum_{m=0}^{\infty}K_{0m}T_{0m}\left\{\frac{[(a/b)^m - (a/b)^{P_{02}-1}]R^{P_{01}-1} + [(a/b)^{P_{01}-1} - (a/b)^m]R^{P_{02}-1}}{(a/b)^{P_{01}-1} - (a/b)^{P_{02}-1}} - R^m\right\} \tag{7a}$$

$$S_\theta{}^0(R) = \sum_{m=0}^{\infty}K_{0m}T_{0m}\left\{\frac{k_{01}[(a/b)^m - (a/b)^{P_{02}-1}]R^{P_{01}-1} + k_{02}[(a/b)^{P_{01}-1} - (a/b)^m]R^{P_{02}-1}}{(a/b)^{P_{01}-1} - (a/b)^{P_{02}-1}} - (m+1)R^m\right\} \tag{7b}$$

$$S_{r\theta}{}^0(R) = 0 \tag{7c}$$

where

$$K_{0m} = \frac{(m+1)\lambda_2c_{11} - [(m+1)\lambda_1 - \lambda_2]c_{12} - \lambda_1c_{22}}{(m+1)^2c_{11} - c_{22}} \tag{7d}$$

$$k_{0j} = (c_{12}P_{0j} + c_{22})/(c_{11}P_{0j} + c_{12}), \qquad j = 1, 2 \tag{7e}$$

The previous stress formulas are valid for $c_{22}/c_{11} \neq (m+1)^2$. When $c_{22}/c_{11} = (m+1)^2$, the corresponding particular solution term for Eq. (6a) has the form $R^{m+1}\log R$ times a constant. The most obvious condition under which this will occur is for an isotropic material in which $c_{11} = c_{22}$ and when $m = 0$. Since the isotropic material has been treated extensively by Forray,[1-3] it will be assumed in all that follows that $c_{11} \neq c_{22}$. The fact that c_{11} and c_{22} will be experimentally determined constants and subject to all the usual limitations thereof, it will be assumed that $c_{22}/c_{11} \neq (m+1)^2$ and formulas for this special case are not presented. It can be shown from the positive definiteness of the strain energy function[5] that $P_{01} \neq -c_{12}/c_{11}$ and $P_{02} \neq c_{12}/c_{11}$.

Solution for n = 1

The U_1 and V_1 which are required to satisfy the field Eqs. (2) have the form

$$U_1(R) = A_{11}R^{P_{11}} + A_{12}R^{P_{12}} + \sum_{m=0}^{\infty}D_{1m}T_{1m}R^{m+1}$$

$$V_1(R) = h_{11}A_{11}R^{P_{11}} + h_{12}A_{12}R^{P_{12}} + n\sum_{m=0}^{\infty}E_{1m}T_{1m}R^{m+1} \tag{8a}$$

$$D_{nm} = \frac{(m+1)n^2\lambda_2c_{12} - (m+1)n^2\lambda_1c_{22} + m[(m+1)(m+2)\lambda_1 + (n^2-m-2)\lambda_2]c_{44}}{c_{11}c_{44}(m+1)^4 + [n^2(c_{12}^2 + 2c_{12}c_{44} - c_{11}c_{22}) - (c_{11}+c_{22})c_{44}](m+1)^2 + (n^2-1)^2c_{22}c_{44}} \tag{8b}$$

$$E_{nm} = \frac{-(m+1)^2\lambda_2c_{11} + (m+1)[(m+1)\lambda_1 - \lambda_2]c_{12} + (m+1)\lambda_1c_{22} + [(m+1)(m+2)\lambda_1 + (n^2-m-2)\lambda_2]c_{44}}{c_{11}c_{44}(m+1)^4 + [n^2(c_{12}^2 + 2c_{12}c_{44} - c_{11}c_{22}) - (c_{11}+c_{22})c_{44}](m+1)^2 + (n^2-1)^2c_{22}c_{44}} \tag{8c}$$

where

$$h_{nj} = \frac{c_{11}P_{nj}{}^2 - (c_{22} + n^2c_{44})}{n[c_{22} + c_{44} - (c_{12} + c_{44})P_{nj}]} \tag{8d}$$

$$P_{11} = -P_{12} = [(1 + c_{22}/c_{11}) + (c_{22}/c_{44}) - (2c_{12}/c_{11}) - (c_{12}^2/c_{11}c_{44})]^{1/2} \tag{8e}$$

and A_{11}, A_{12} are arbitrary constants to be determined from the boundary conditions. After evaluation, the R dependent terms of Eqs. (4b) are found to be

$$S_r{}^1(R) = S_{r\theta}{}^1(R) = \sum_{m=0}^{\infty}K_{1m}T_{1m}\left\{\frac{[(a/b)^m - (a/b)^{P_{12}-1}]R^{P_{11}-1} + [(a/b)^{P_{11}-1} - (a/b)^m]R^{P_{12}-1}}{(a/b)^{P_{11}-1} - (a/b)^{P_{12}-1}} - R^m\right\} \tag{9a}$$

$$S_\theta{}^1(R) = \sum_{m=0}^{\infty}K_{1m}T_{1m}\left\{\frac{k_{11}[(a/b)^m - (a/b)^{P_{12}-1}]R^{P_{11}-1} + k_{12}[(a/b)^{P_{11}-1} - (a/b)^m]R^{P_{12}-1}}{(a/b)^{P_{11}-1} - (a/b)^{P_{12}-1}} - (m+2)R^m\right\} \tag{9b}$$

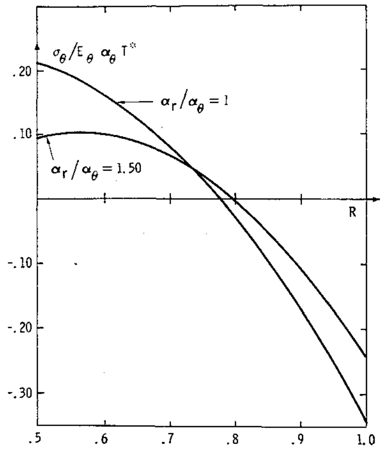

Fig. 1 Circumferential stress.

where

$$K_{1m} = \frac{c_{44}[m\lambda_2 c_{11} - (m\lambda_1 - \lambda_2)c_{12} - \lambda_1 c_{22}]}{c_{11}c_{44}(m+1)^2 + c_{12}^2 + 2c_{12}c_{44} - c_{11}c_{22} - c_{11}c_{44} - c_{22}c_{44}} \quad (9c)$$

$$k_{1j} = \frac{(c_{11}c_{22} - c_{12}^2 - c_{12}c_{44})P_{1j} + (c_{12} - c_{22})c_{44}}{-c_{11}c_{44}P_{1j} + c_{11}c_{22} - c_{12}^2 + (c_{11} - c_{12})c_{44}} \qquad j = 1,2 \quad (9d)$$

The denominator of K_{1m} has a zero when $m + 1 = P_{11}$ or $m + 1 = P_{12}$. When this condition occurs, either a different particular solution term must be found or a suitable limiting process invoked. This situation implies a specific relation among the elastic constants, and since these are experimentally determined as discussed previously for the case $n = 0$, it will be assumed that this required relationship is not met. Also, h_{11} and k_{11} can be indeterminant, but again, this requires that the elastic constants satisfy a special relation.

Solution for $n \geq 2$

For $n \geq 2$ the R dependent terms of Eqs. (4a) take the form

$$U_n(R) = \sum_{j=1}^{4} A_{nj}R^{P_{nj}} + \sum_{m=0}^{\infty} D_{nm}T_{nm}R^{m+1}$$
$$V_n(R) = \sum_{j=1}^{4} h_{nj}A_{nj}R^{P_{nj}} + n\sum_{m=0}^{\infty} E_{nm}T_{nm}R^{m+1} \quad (10)$$

and the associated $S_r{}^n(R)$, $S_\theta{}^n(R)$, and $S_{r\theta}{}^n(R)$ may be obtained from Eqs. (1). In Eq. (10), the P_{nj} are the four roots of

$$P_{nj}{}^4 + \frac{[n^2(c_{12}^2 + 2c_{12}c_{44} - c_{11}c_{22}) - (c_{11} + c_{22})c_{44}]}{c_{11}c_{44}} P_{nj}{}^2 + (n^2 - 1)^2 \frac{c_{22}}{c_{11}} = 0 \quad (11)$$

and the A_{nj} are arbitrary constants. The traction free boundary conditions at $R = a/b$ and $R = 1$ render four simultaneous equations for the determination of the A_{n1}, A_{n2}, A_{n3} and A_{n4}. The solution of these equations is quite lengthy and will not be presented. It should be noted here that the P_{nj} may be real or complex. If they are complex, they occur as complex conjugate pairs and the associated A_{nj} must also be complex conjugate pairs to insure that the displacements and stresses be real.

Summary and Numerical Results

Formulas for the thermal stresses in a transversely isotropic, hollow circular cylinder have been derived for the temperature distribution $T(R, \theta)$ given by Eq. (3). Explicit formulas are presented for $n = 0$ in Eqs. (7) and for $n = 1$ in Eqs. (9); results for higher harmonics can be obtained from Eqs. (10).

Circumferential stress components for a cylinder with $a/b = 1/2$, $E_r/E_\theta = 0.970$, $G_{r\theta}/E_\theta = 0.500$, $\nu_{z\theta} = \nu_{\theta r} = 0.10$ are presented in Fig. 1 for $\alpha_r/\alpha_\theta = 1.00$, 1.50, and the temperature distribution $T = T^*(1 - 2R + 2R^2)$. These material properties are typical for ATJ graphite. As previously mentioned, the Young's moduli and Poisson's ratios are related to c_{ij}; e.g., see Ref. 4.

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Laminar Reynolds Analogy Factor on a Sharp Flat Plate at Mach 10.4

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THERE are several theoretical variations of Reynolds analogy factor ($2N_{St}/C_f$), but a simple and often used value at hypersonic speeds is the Colburn value of the Prandtl number to the $-\frac{2}{3}$ power. Only a few published studies, however, attempt to determine the validity of this value under various flow conditions. It is the purpose of this Note to present some values measured on a sharp flat plate in a Mach 10.4 boundary-layer induced-pressure gradient and to show, by theoretical calculations, the effect of the wall-pressure and wall-temperature gradients that are present in the experiment.

Two 30-in.-long sharp-edge flat plates of aspect ratio 0.5 with identical exterior dimensions were tested in the continuous flow hypersonic tunnel at the NASA Langley Research Center. The thinner plate (0.031 in. thick) was instrumented with thermocouples to obtain heat transfer by the transient calorimeter technique. The thicker plate (0.187 in. thick) was instrumented to obtain the wall-pressure distribution and was used with traversing probes to obtain boundary-layer impact pressure profiles. The heat-transfer test interval was about five seconds whereas the wall-pressure measurements and boundary-layer profiles required about two minutes of exposure to the hypersonic stream. A more complete description of the models and tests and the resultant velocity profiles and heat-transfer distributions are presented in Ref. 1.

The theoretical approach used to calculate wall-pressure gradient and wall-temperature gradient effects was an implicit finite difference solution to the compressible laminar boundary-layer equations for a perfect gas and a Sutherland

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